Calculating the Transition Energy, γ_T, at RHIC

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RHIC is reported to be the first heavy ion machine to go through "transition" using super-conducting magnets. This is a difficult because it requires ramping the magnets quickly to avoid beam instabilities … and a fast ramp is hard to do with super-conducting magnets.

So what is transition? Why does it cause beam instabilities? And why is $\gamma_T \approx 23$ at RHIC?

In a linear accelerator, the beam is bunched into packets and the fastest particles are at the front of the pack and the slowest particles are at the back. So, for example, the bunch length can be reduced by slowing down the leading particles and speeding up the trailing particles with a pulse of RF that has a wavelength and phase to match the packets velocity and bunch length.

A circular accelerator is more complex. The beam packets travel in a closed loop under the influence of a magnetic field that causes the fastest particles in the bunch to travel in slightly larger radius orbits than the slowest particles in the bunch. Thus, the fastest particles can take more time to go around the ring than the slowest particles due to the longer flight path for the faster particles. However, an accelerator is composed of many magnets and many straight sections and in the straight sections, the faster particles take less time to traverse the straight path. This sets up the possibility of designing an accelerator where all energies take equal time to go through the magnets and straight sections because the extra time lost by the fast particles going through the magnets can be regained in the straight sections.

This is not useful. A beam bunching system, for example, must operate on the different energy particles in the packet at different times – such as de-accelerating the leading particles and later accelerating the trailing particles. If all energies arrive at the same time, then a bunching element cannot act on the beam in a useful way and the beam can't be controlled – and so it eventually it becomes unstable.

One way to store a beam in a circular accelerator is to have the fastest particles in a bunch traveling at large radii and leading the slowest particles at small radii. It is also possible to store a stable beam in the opposite configuration: with the fastest particles at large radii … but trailing the slowest particles which are orbiting at smaller radii. Either configuration is possible but it depends on the geometry of the machine *and* the energy of the beam to define a stable configuration.

"Transition" is the configuration where the slowest and fastest particles in a bunch circulate through the machine isochronously – neither the fast particles nor the slow particles are the leading particles. They go through the machine in equal time but on different radius orbits. In other words, "transition" is the unstable point between the two beam configurations described above and the transition energy is the beam energy when it happens.

Example I: Particles in a dipole field

Assume a relativistic particle in a very large dipole magnet. The relationship between the momentum of the particle and the radius of its orbit is:

(1) $p \; = \; 0.3\, B\, z\, R$ [Tesla meters / GeV / c]

The time to complete an orbit is:

(2) $\tau = D / V$

where D = distance and V = the velocity of the beam.

$$\tau \; = \; 2\pi R\, \gamma\, m \,/\, p \qquad \Rightarrow \qquad 2\pi\, \gamma\, m \,/\, 0.3\, B\, z$$

This result says that high momentum particles take *more* time to complete an orbit than low momentum particles. There is no maximum, no minimum, and no transition. This behavior is due to the larger radius of orbit in the magnets for the high momentum particles and the fact that all velocities are bunched up near the speed of light.

Example II: Particles in a storage ring

A simple storage ring is composed of dipole magnets and straight sections arranged to form a closed loop. And we showed, above, that high energy, or high momentum, particles take more time to travel through the dipole magnets than the low energy particles. The reverse is true in the straight sections: the high energy particle take less time to travel through the field free region of pipe than the low energy particles. So the total time to orbit in the storage ring depends on the ratio of the path length in the magnetic field to the path length in the straight sections.

The RHIC ring is made of 6 arcs and 6 straight sections. The orbital period is:

(3) $$\tau \; = \; 2\pi R \,/\, V \; + \; 6L \,/\, V \qquad \Rightarrow \qquad 2\pi\, \gamma\, m \,/\, 0.3 B\, z \; + \; 6L \,/\, c \sqrt{1 - {}^{1}\!/_{\gamma^2}}$$

where R is the radius of curvature of the particle in the dipole field and L is the length of a straight section. Equation 3 has a minimum that can be found by differentiating it as a function of γ:

$$\frac{d\tau}{d\gamma} \; = \; 2\pi\, m \,/\, 0.3 B\, z \; - \; 6L \,/\, c \left(\gamma^2 - 1\right)^{3/2} \qquad \Rightarrow \qquad 2\pi R\, m \,/\, p \; - \; 6L \; \left(mc^2 / pc\right)^3 / \, c$$

and setting this to zero yields

$$\left(pc/mc^2\right)^2 \;=\; \frac{6L}{2\pi R} \qquad \text{or} \qquad \gamma^2 \;=\; 1 + \frac{6L}{2\pi R}$$

At RHIC, L is approximately 60% of R and so the minimum time to orbit the storage ring occurs at $\gamma_T \approx 1.25$

Thus, the competition between the time of flight through the magnets versus the straight sections dictates that high energy particles in a beam packet will lead the lower energy particles around the ring for all packets with average energies $\gamma_T \leq 1.25$; while the low energy particles will lead the high energy particles for all beam packets with average energies above $\gamma_T \geq 1.25$. The energy where all particles in a beam packet orbit the ring in equal time is the transition energy. It is an unstable beam condition because the beam conditioning elements (e.g. bunching and acceleration) cannot simultaneously re-bunch or accelerate all beam momenta. There must be a time delay between the leading and trailing particles for the RF driven elements to be able to apply a different correction to each group.

Unfortunately, the transition energy at RHIC is not $\gamma_T = 1.25$ It is larger. So what went wrong with our calculation?

Consider equation 1. Differentiate it and re-arrange terms to find

(4) $$\frac{dp}{p} = \frac{dR}{R}$$

Equation 4 says that the natural momentum spread of particles in the beam packets is directly translated into a change in orbit radius. And at RHIC the characteristic radius of curvature in the magnets is 380 meters while the characteristic momentum spread of the beam is 0.1% (going up to about 0.5% after a 10 hour store). This means the packets are smeared over 38 centimeters in radius … and the amount of space required increases the longer the beam is stored in the ring.

The beam pipe isn't this big! So we need to find a way to compress the beam into a smaller pipe.

(5) $$\frac{dp}{p} = \alpha \frac{dR}{R}$$

In other words, we need a magnet structure to bend the beam but without smearing it so badly, or equivalently, we need a magnet where α is large. How large? Well, at RHIC, the goal is to keep the beams confined to a millimeter cross section. So α should be roughly 380.

What's the trick? At an accelerator like the Alternating Gradient Synchrotron (AGS), the trick is called strong focusing. The "dipoles" are not standard dipole magnets. The pole tips are tilted slightly so that the magnetic field at small radius is weaker than the field at large radius. This causes the low momentum particles to have a larger radius of curvature and the high momentum particles to have a lesser radius. If done properly, a dipole magnet with tilted pole tips will keep the lower momentum particles at smaller radii than the high momentum particles; but the range of radii is compressed when compared to a regular dipole magnet. And, in fact, this in only part of the story regarding strong focusing but it is sufficient for our purposes. The bend magnets at the AGS are described by equation 5. The magnets at RHIC are also described by equation 5, but the trick is a little different. Standard dipole magnets are used together with quadrupole focusing magnets to achieve the momentum compaction factor, α, in equation 5. RHIC doesn't use "strong focusing magnets" but, rather, it simulates strong focusing with a combination of quadrupoles and dipoles.

Now we can proceed to calculate the orbital period by making a Taylor expansion about the mean orbit radius:

$$\tau \;=\; 2\pi\left(R_0 + (p-p_0)\frac{dR}{dp}\right)/V \;+\; 6L/V \;\Rightarrow\; \frac{2\pi R_0 + 6L - 2\pi R_0/\alpha}{V} \;+\; \frac{2\pi R_0\,\gamma\;m}{\alpha\,p_0}$$

$$\Rightarrow\; \frac{2\pi R_0 + 6L - 2\pi R_0/\alpha}{c\sqrt{1-\frac{1}{\gamma^2}}} \;+\; \frac{2\pi R_0\,\gamma\;m}{\alpha\,p_0}$$

and

$$\frac{d\tau}{d\gamma} \;=\; (-1)\frac{2\pi R_0 + 6L - 2\pi R_0/\alpha}{c\left(\gamma^2-1\right)^{3/2}} \;+\; \frac{2\pi R_0\;m}{\alpha\,p_0}$$

thus, the minimum occurs when

$$\frac{2\pi R_0}{\alpha} \;=\; \frac{2\pi R_0 + 6L - 2\pi R_0/\alpha}{\left(\gamma^2-1\right)} \qquad\Rightarrow\qquad \gamma^2 \;=\; \alpha\left(1+\frac{6L}{2\pi R_0}\right)$$

or, in other words,

$$\gamma_T \;\approx\; 24.6$$

Conclusions:

A relativistic particle beam in a circular accelerator must have a transition point and this is due to the momentum compaction factor in the bend magnets and, at RHIC, due to the different amounts of time it takes for the high and low momentum particles to go through the bend magnets relative to the straight sections.

Or, in other words, the reason why there is a transition energy at RHIC is because the beams must fit inside the beam pipes.

Comments:

A transition point exists in a circular accelerator as long as the magnets have a compaction factor greater than 1, even if the drift sections have zero length. The effect of the drift sections, at RHIC, is to increase the transition energy from 18 to 23.

Gamma is imaginary when the compaction factor is negative. A negative compaction factor means that low momentum particles occupy the largest average orbit radius in the bend magnets and thus the low momentum particles are *always* trailing the high momentum particles. Hence, there is no transition to cause beam instabilities. On paper, gamma is "imaginary". Designing the magnets, however, is tricky and this idea introduces other issues into the beam optics design.

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